MORE QUALITATIVE RESULTS FOR THE QUADRATIC FUNCTION APPROXIMATION

by

R.G. Brookes and **A.W. McInnes**

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Abstract

This report continues the examination of the properties of the quadratic approximation in some particular cases. The approximations are shown to be valid over a wider region than is immediately apparent.

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Key words and phrases: Hermite-Padé approximation, quadratic function approximation, qualitative properties of approximation.

1. Introduction

In this report some of the characteristics of the quadratic Hermite-Padé approximation which were addressed in [2] are examined in further detail. In particular it is shown that in the examples studied it is possible to ignore many of the spurious singularities that occur in the approximation. This leads us to examine the approximation over a much larger region than was the case in [2]. The examples chosen for study are $\cos x$, $\log(1+x)$ and $\sqrt[3]{1+x}$. The $(4,4,4)$ approximation is examined in all cases and comparisons made with the appropriate Padé approximations.

2. Examples

Example 1 : $\cos x$

The $(4,4,4)$ approximation to $\cos x$. Note that

(i)

$$\begin{aligned}&\left(11041x^4+953925x^2+30370095\right)f(x)^2\\+&\left(-1196192x^4-134400x^2-324253440\right)f(x)\\+&5459071x^4-132576150x^2+293883345=O\left(x^{16}\right)\end{aligned}$$

so that, using the results of [1], the approximation is

$$y(x)=\frac{-a_1(x)-\sqrt{D(x)}}{2a_2(x)}.$$

This Hermite-Padé form is, in fact, the best choice from the 2 dimensional space of (4,4,4) forms (this topic will be investigated in a further report) so is of greater order of accuracy than might otherwise be expected.

(ii)

$$D(x) = 1189780889220x^8 - 14653547716500x^6 + 605478537140400x^4 + 15071189726092500x^2 + 69439232925562500$$

so the roots of $D(x)$ are
$x = \pm 2.8245\mathbf{i}$
$x = \pm 2.8489\mathbf{i}$
$x = \pm(4.7026 \pm 2.8123\mathbf{i})$.
Note also that the roots of $a_2(x)$ are $x = \pm 2.1503 \pm 6.9154\mathbf{i}$

In a manner similar to that of [2] (Example 2) cuts are taken between the zeroes of small separation, namely $\{x\mathbf{i} : x \in (-2.8489, -2.8245)\}$ and $\{x\mathbf{i} : x \in (2.8245, 2.8489)\}$. Graphs and contour maps of the error function $e(z) = y(z) - \cos(z)$ drawn with PC-Matlab are now given. The region shown is $\{x + \mathbf{i}y \in \mathbf{C} : |x|, |y| \leq 5\}$ with a mesh spacing of 0.25 and $e(z)$ has been truncated at ± 1.

Figures 1 and 2 are graphs of real($e(z)$). Figures 3 and 4 are contour maps of real($e(z)$) and imag($e(z)$) with contours drawn at $\{\pm 1, \pm 10^{-1}, \pm 10^{-2}, \ldots, \pm 10^{-5}\}$.

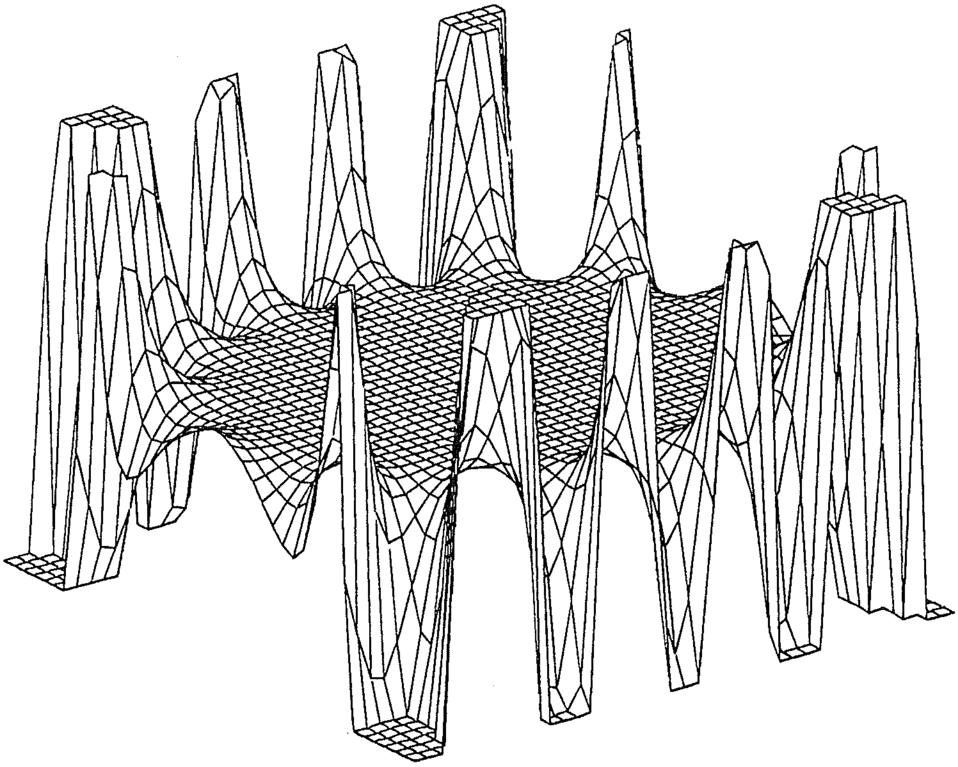

Figure 1 Real(e(z)). Truncation ± 1

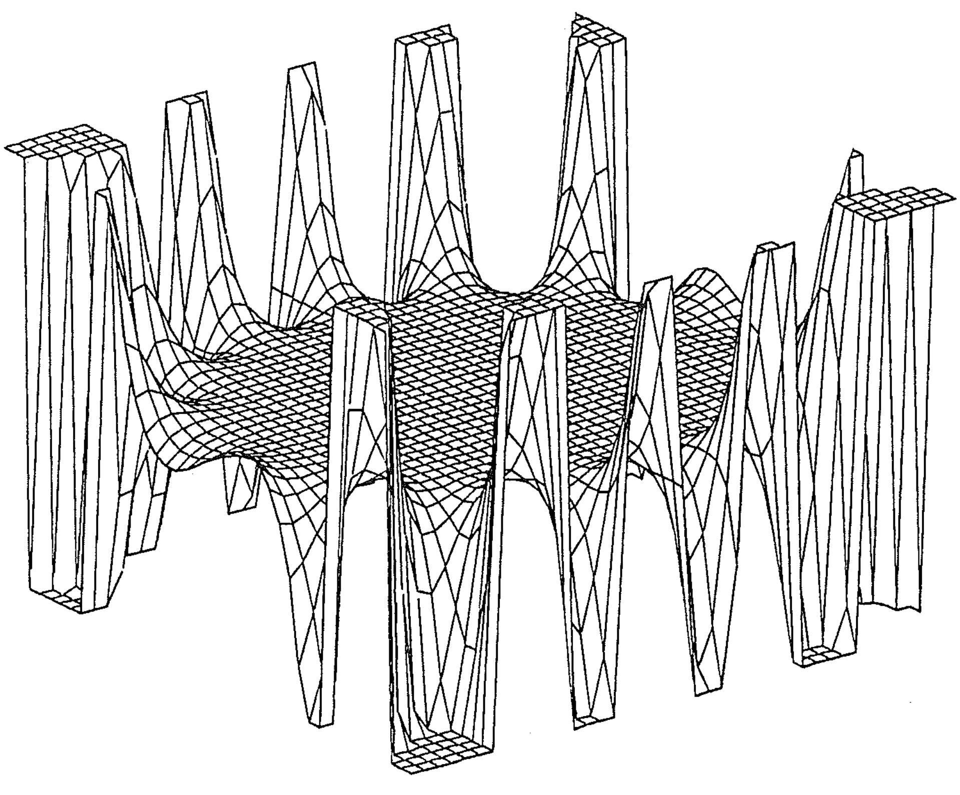

Figure 2 Imag(e(z)). Truncation ± 1

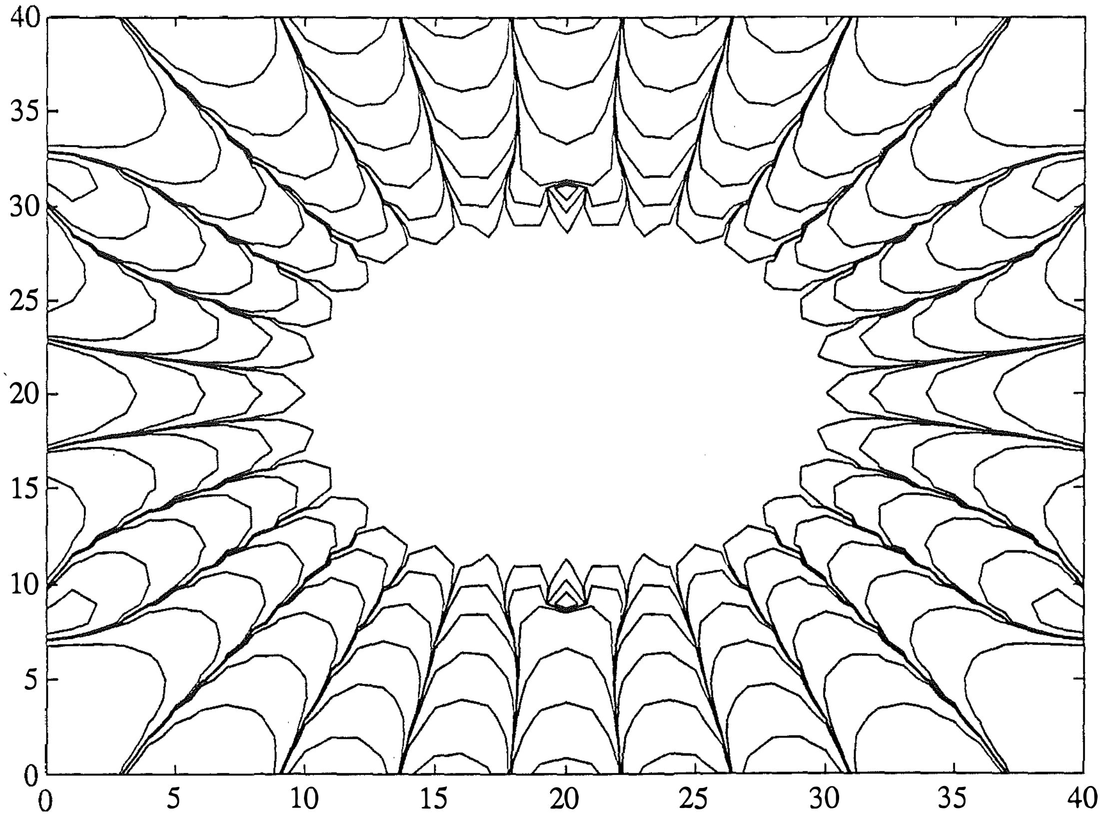

Figure 3 Contour map of Real(e(z)).

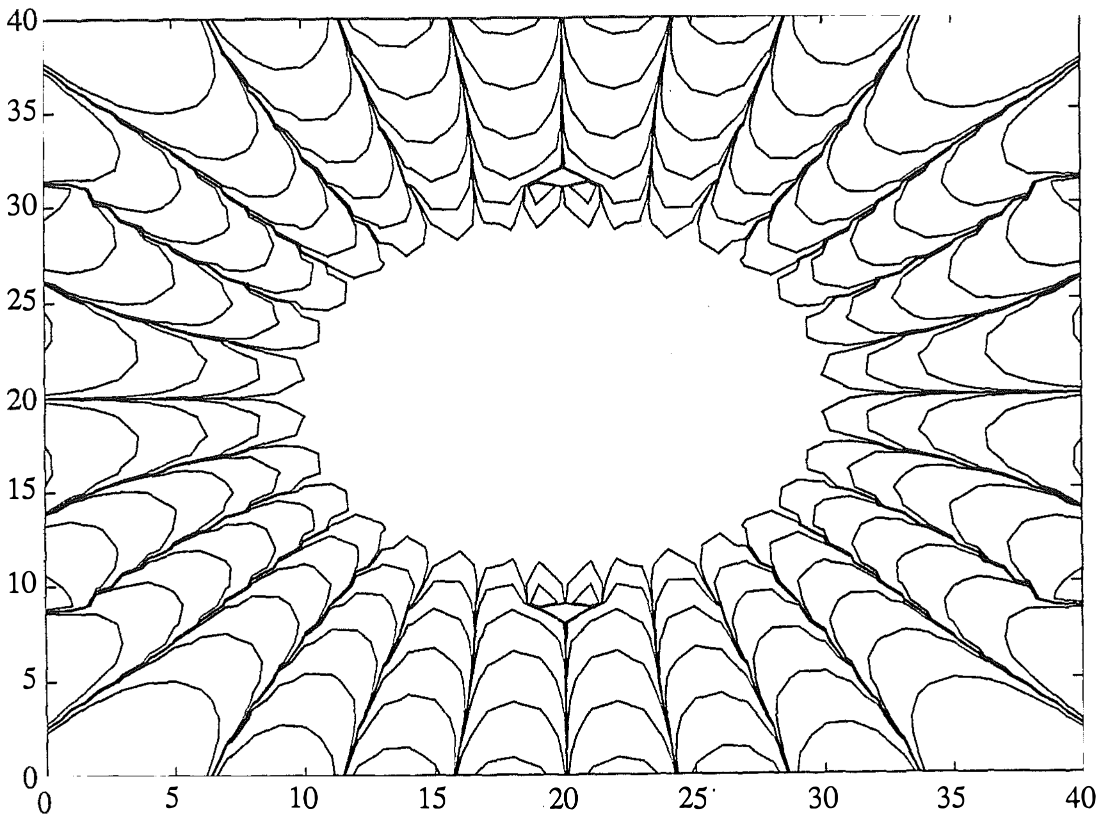

Figure 4 Contour map of Imag(e(z)).

The $(6,8)$ **Padé approximation to cos**(x)**.** There is no $(7,7)$ Padé approximtion which matches $f(x)$ up to $O(x^{16})$ so the $(6,8)$ approximation has been chosen instead. Note that

$$p(x) = \frac{45469x^8 - 7029024x^6 + 348731040x^4 - 5269904640x^2 + 10983772800}{9336x^6 + 2064720x^4 + 221981760x^2 + 10983772800}$$

and that

$$\begin{aligned} y(x) &= f(x) + O\left(x^{16}\right) \\ p(x) &= f(x) + O\left(x^{16}\right) . \end{aligned}$$

Figures 5 and 6 are graphs of real$(e(z))$ and imag$(e(z))$ (where $e(z) = p(z) - \cos(z)$) truncated at ± 1. Figures 7 and 8 are contour maps of real$(e(z))$ and imag$(e(z))$ with contours drawn at $\{\pm 1, \ldots, \pm 10^5\}$.

It is clear that $y(x)$ is much superior to $p(x)$ as an approximation to cos(x) over a considerably wider area.

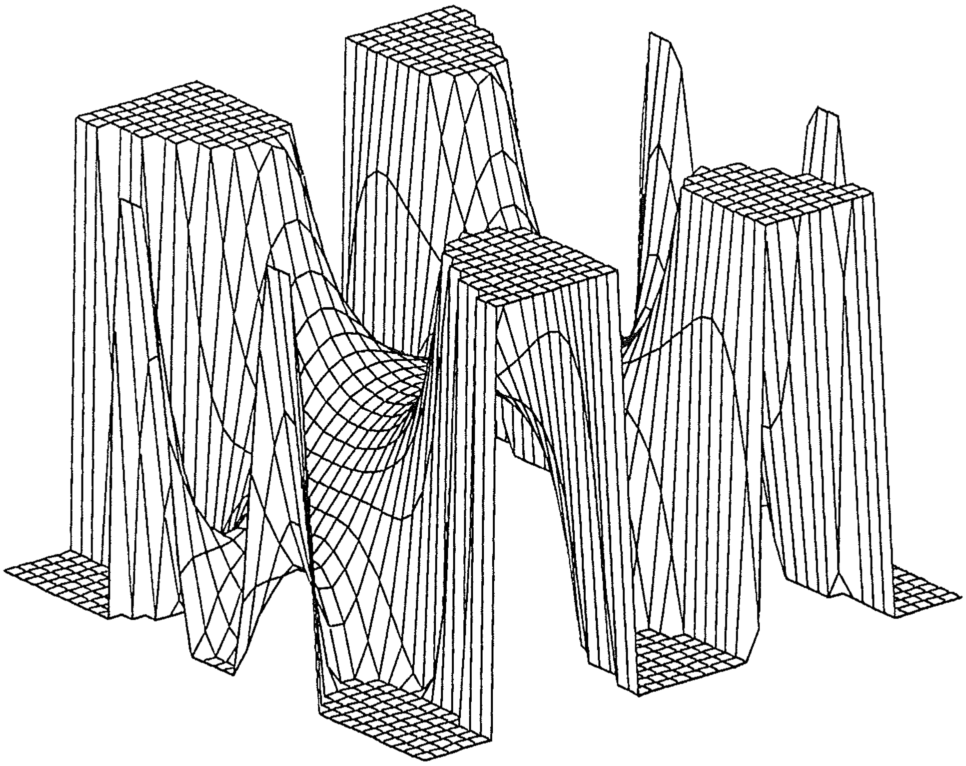

Figure 5 Real(e(z)). Truncation ± 1

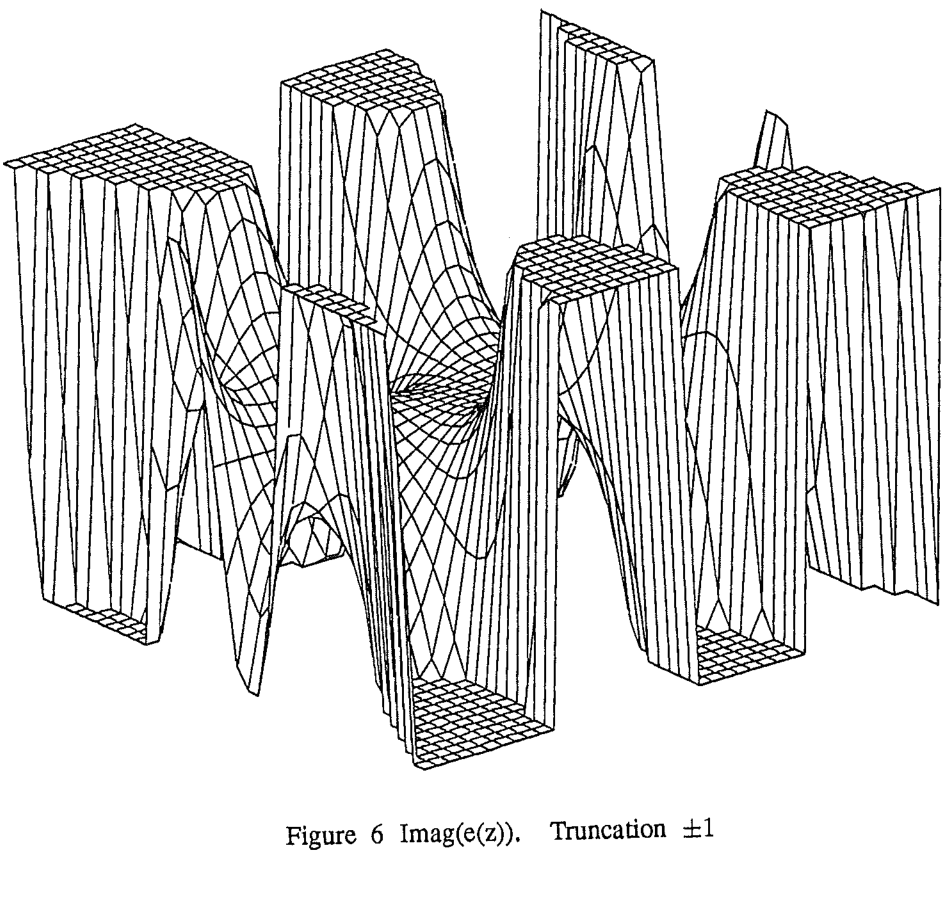

Figure 6 Imag(e(z)). Truncation ± 1

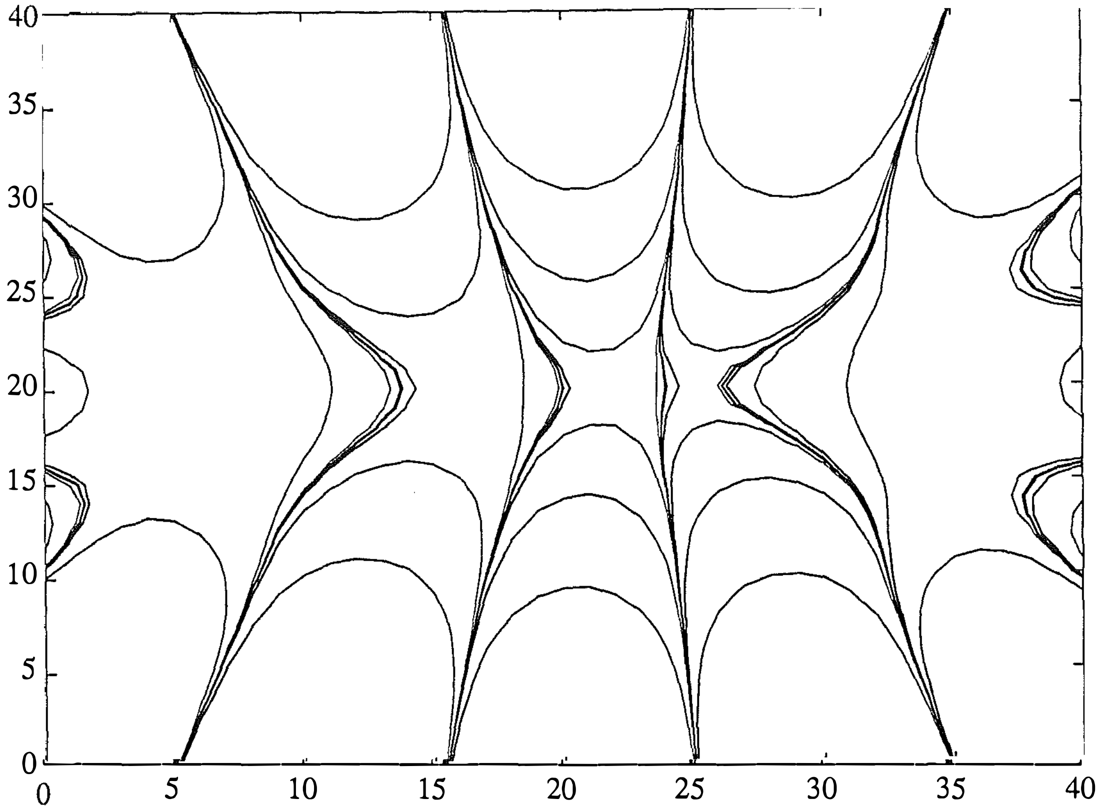

Figure 7 Contour map of Real(e(z)).

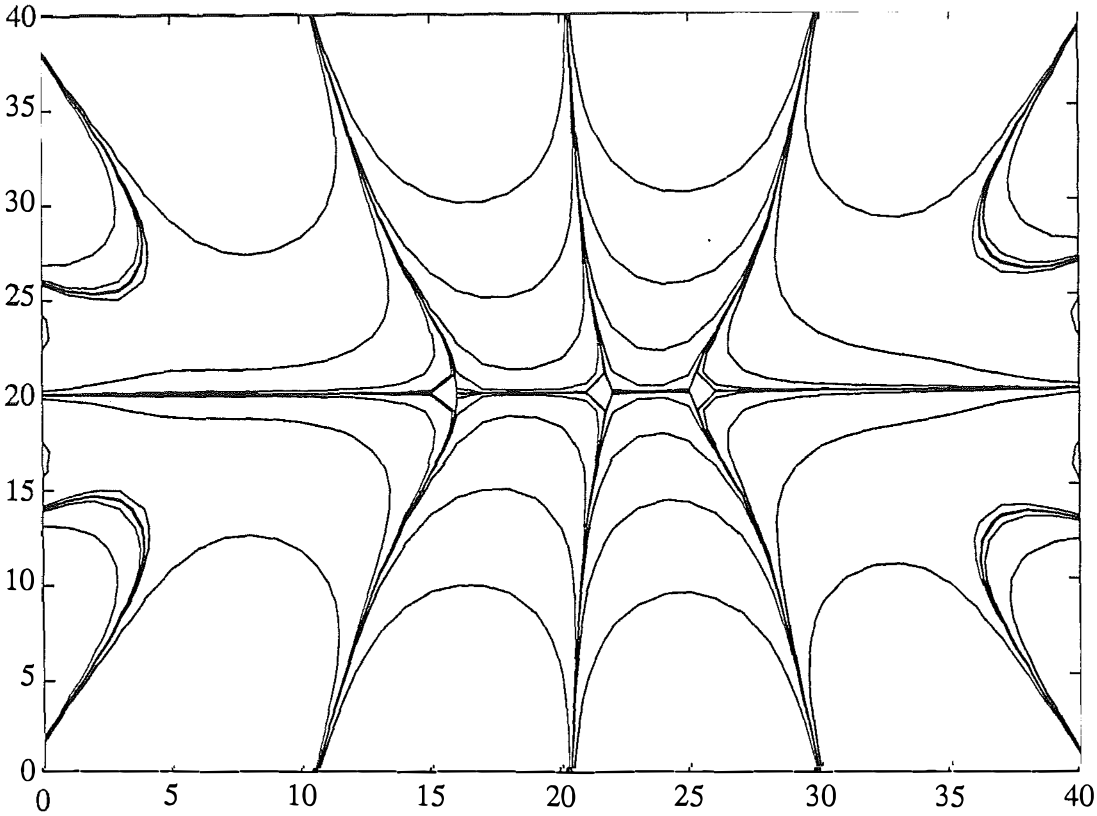

Figure 8 Contour map of Imag(e(z)).

Example 2 : $\log(1+x)$.

The $(4,4,4)$ approximation to $\log(1+x)$. Note that (see [2], 3.2.1)

(i)

$$\begin{aligned} &\left(6x^4 - 360x^3 + 180x^2 + 1080x + 540\right) f(x)^2 \\ &+ \left(-75x^4 + 1620x^3 + 5310x^2 + 3540x\right) f(x) \\ &+260x^4 - 4080x^3 - 4080x^2 = O\left(x^{14}\right) \end{aligned}$$

(ii) $y(x) = \dfrac{-a_1(x) + x\sqrt{d(x)}}{2a_2(x)}$ where

$$\begin{aligned} d(x) = &- 615x^6 + 229320x^5 - 4136580x^4 \\ &+ 12612600x^3 + 59667300x^2 + 64033200x \\ &+ 21344400\,. \end{aligned}$$

The roots of $d(x)$ are:

$$\begin{aligned} x &= 354.0459 \\ x &= 10.8301 \pm 0.06444\mathbf{i} \\ x &= -0.9155 \pm 0.0005\mathbf{i} \\ x &= -0.9972\,. \end{aligned}$$

In [2] (3.2.1) it was shown that by defining a cut $\{x + \mathbf{i}y \in \mathbf{C} : x = -0.9155, |y| \leq 0.0005\}$ a good approximation to $\log(1+z)$ was obtained. The conjugate roots $z = 10.8301 \pm 0.06444\mathbf{i}$ can be treated in the same way. A cut has been defined as $\{x + \mathbf{i}y \in \mathbf{C} : x = 10.8301, |y| \leq 0.06444\mathbf{i}\}$ and then the error function $e(z)$ graphed on the region $\{x + \mathbf{i}y \in \mathbf{C} : |x|, |y| \leq 20\}$ with a mesh spacing of 1.

Figures 9 and 10 are graphs of real$(e(z))$ and imag$(e(z))$ truncated at ± 1. Figures 11 and 12 are contour maps of real$(e(z))$ and imag$(e(z))$ with contours drawn at $\{\pm 1, \pm 10^{-1}, \ldots, \pm 10^{-5}\}$.

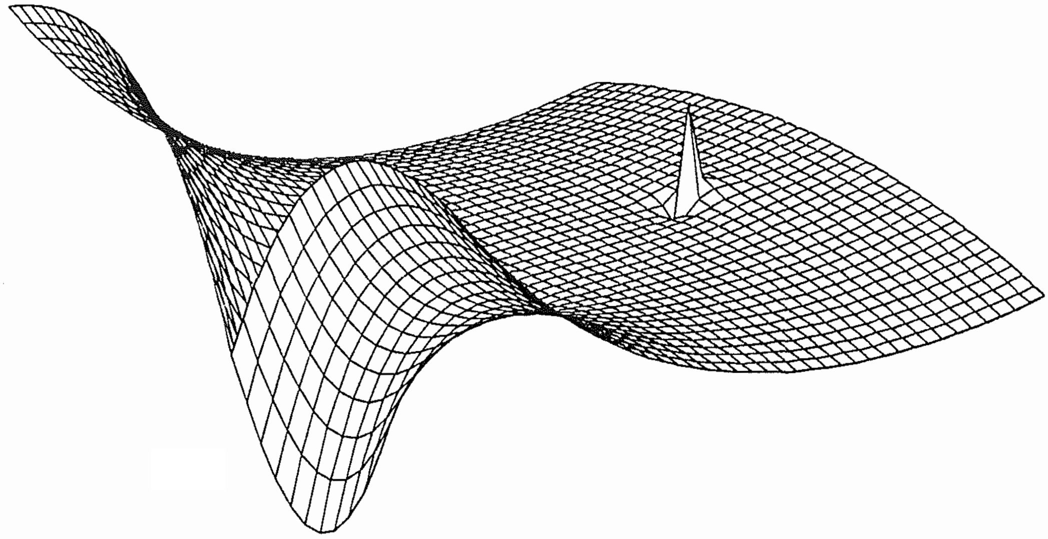

Figure 9 Real(e(z)). Truncation ± 1

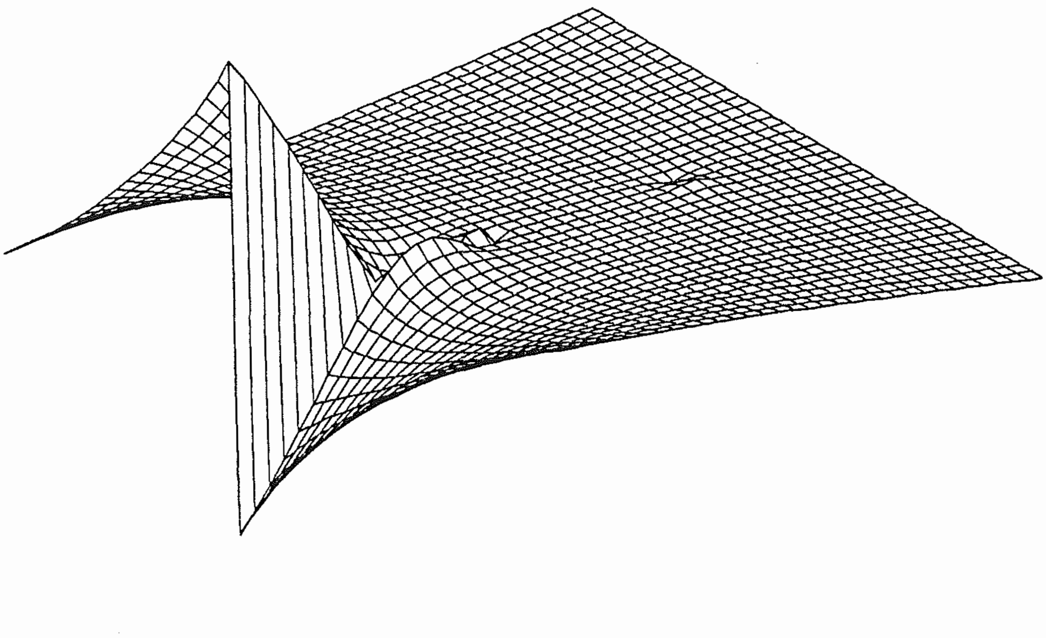

Figure 10 Imag(e(z)). Truncation ± 1

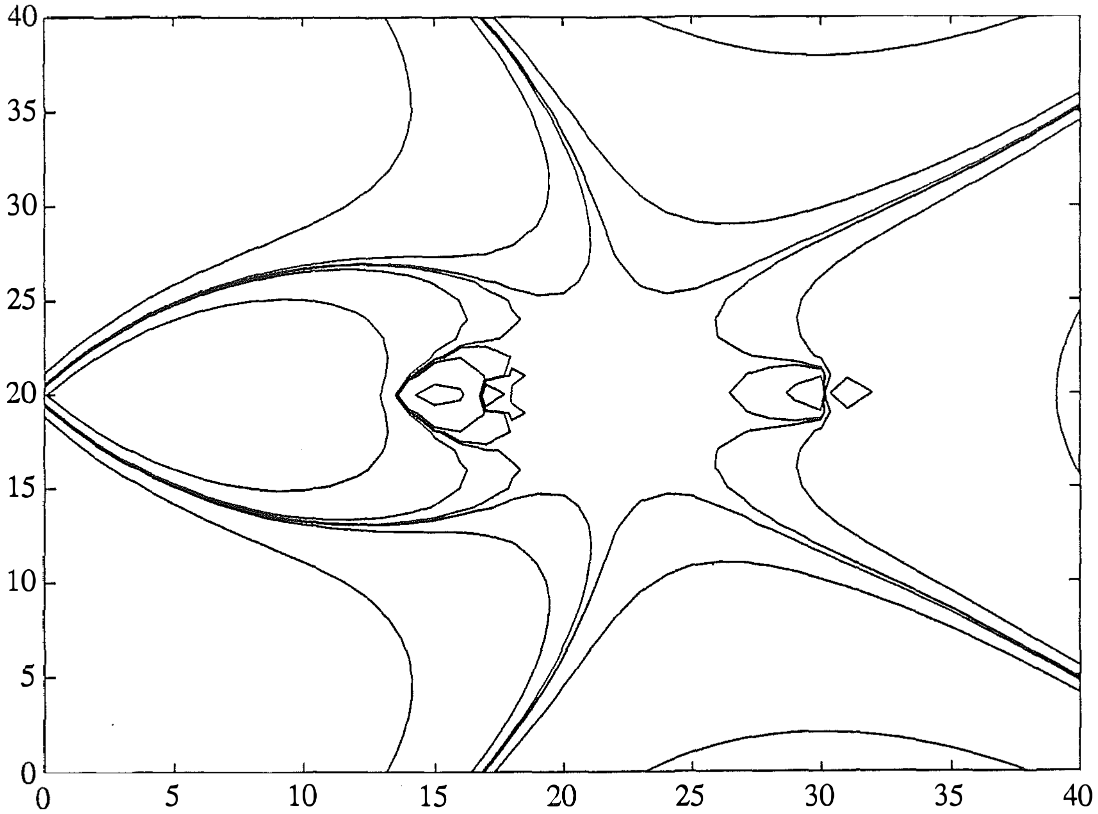

Figure 11 Contour map of Real(e(z)).

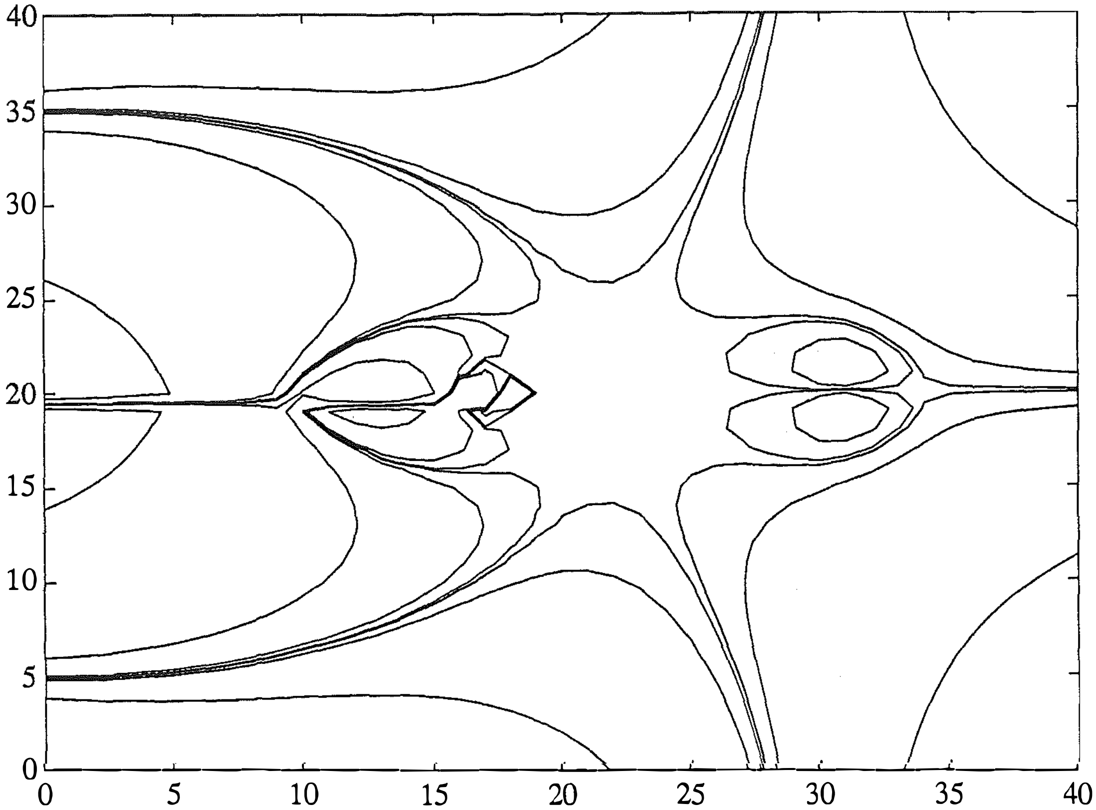

Figure 12 Contour map of Imag(e(z)).

The $(6,6)$ **Padé approximation to** $\log(1+x)$**.** Note that

$$p(x) = \frac{49x^6 + 1218x^5 + 7980x^4 + 20720x^3 + 23100x + 9240x}{10x^6 + 420x^5 + 4200x^4 + 16800x^3 + 31500x^2 + 27720x + 9240}$$

and that

$$\begin{aligned} y(x) &= f(x) + O\left(x^{13}\right) \\ p(x) &= f(x) + O\left(x^{13}\right) . \end{aligned}$$

Figures 13 and 14 are graphs of real$(e(z))$ and imag$(e(z))$ truncated at ± 1. Figures 15 and 16 are the usual contour maps of real$(e(z))$ and imag$(e(z))$.

Clearly $p(x)$ is inferior to $y(x)$ as an approximation to $\log(1+x)$ in this region. As a further illustration a graph showing $p(x), y(x)$, $\log(1+x)$along the positive real axis from 0 to 350 is given in Figure 17. Here $y(x)$ is represented by a solid line, $\log(1+x)$ by "– –" and $p(x)$ by "$\cdot\cdot\cdot$". $y(x)$ is reasonably accurate and certainly superior to $p(x)$ out to this point. As one approaches the branch point of $y(x)$ at $x = 354.0459$ the performance of $y(x)$ deteriorates. Beyond the branch point, however, real$(y(x))$ is still a good approximation.
e.g.

$$\begin{aligned} f(1000) &= 6.91 \\ real(y(1000)) &= 6.50 \\ p(1000) &= 4.82 \end{aligned}$$

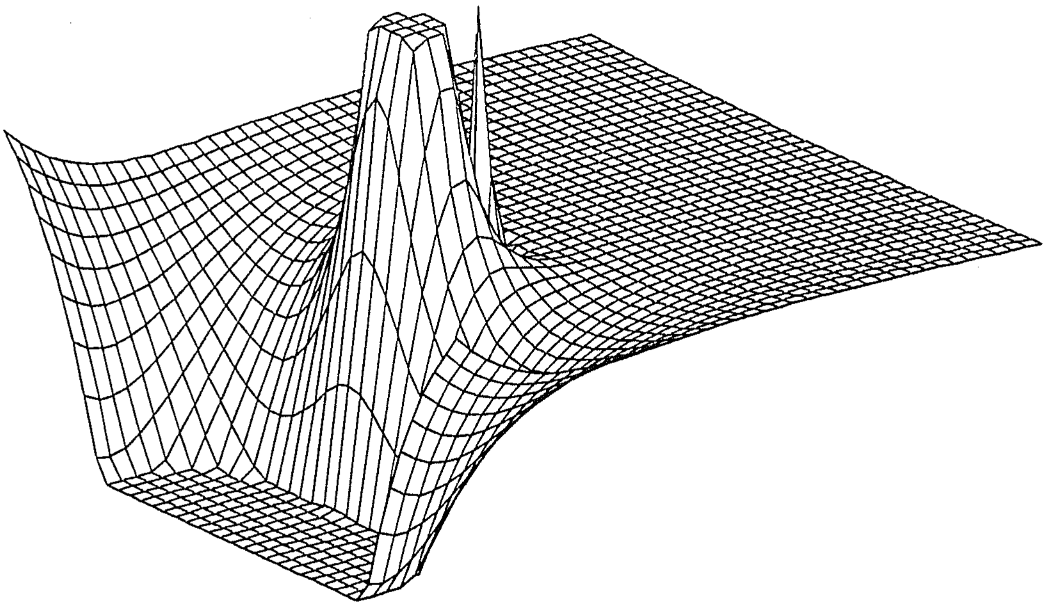

Figure 13 Real(e(z)). Truncation ± 1

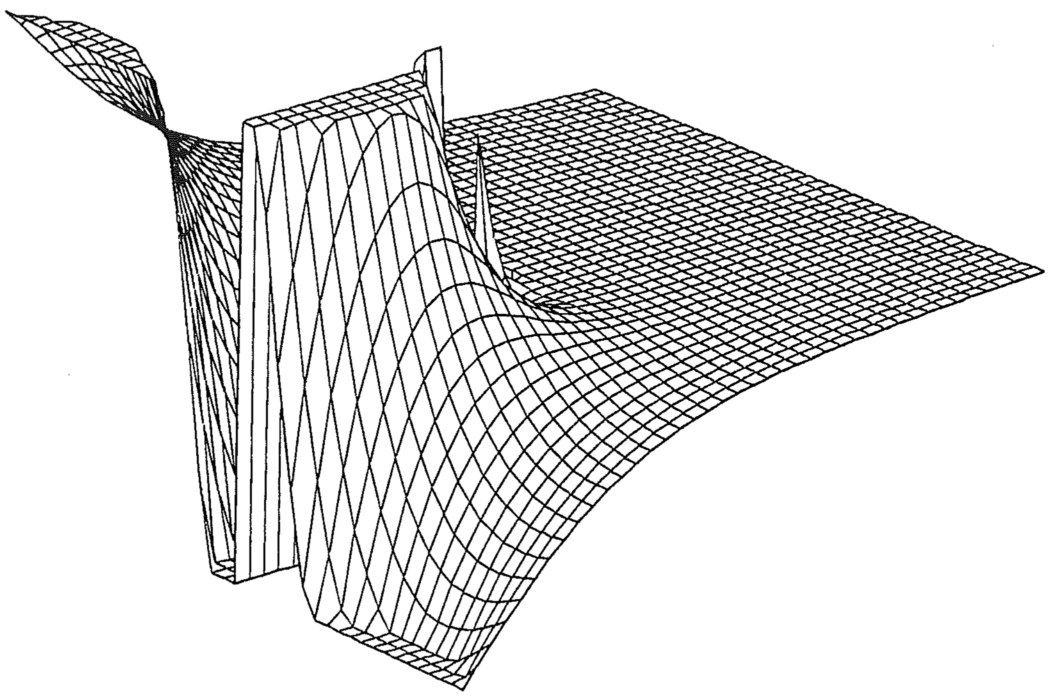

Figure 14 Imag(e(z)). Truncation ± 1

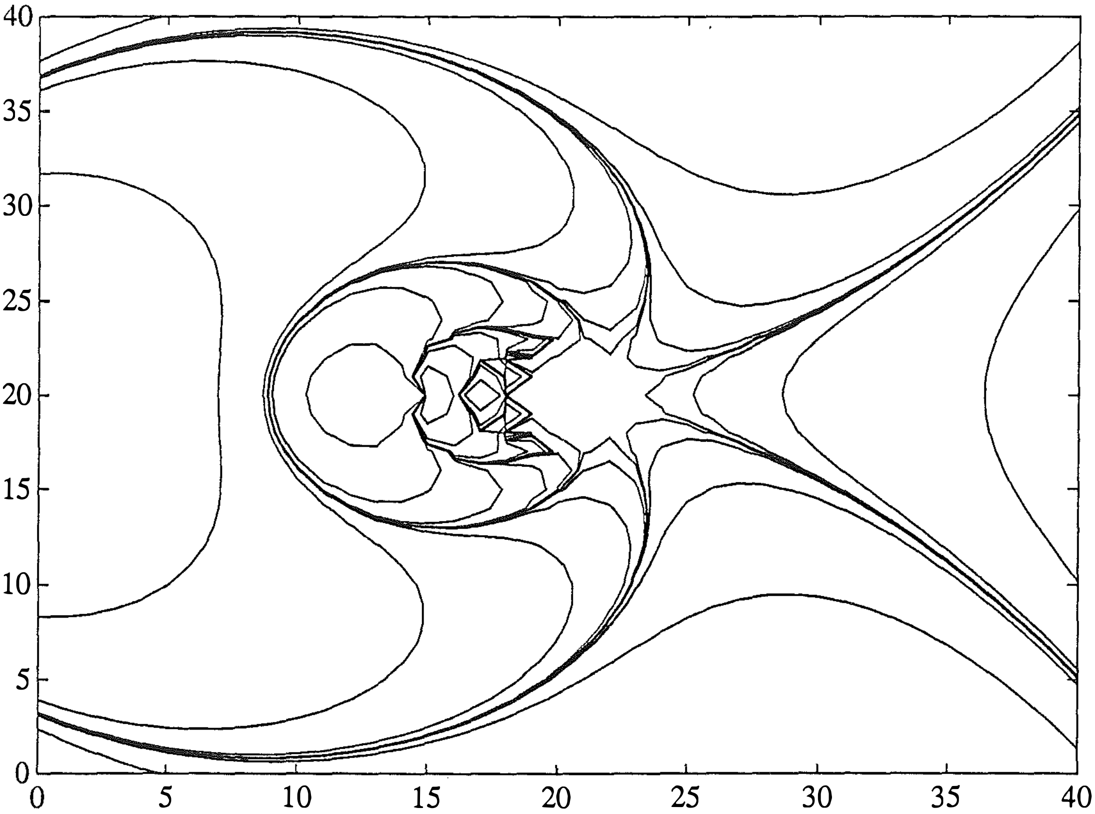

Figure 15 Contour map of Real(e(z)).

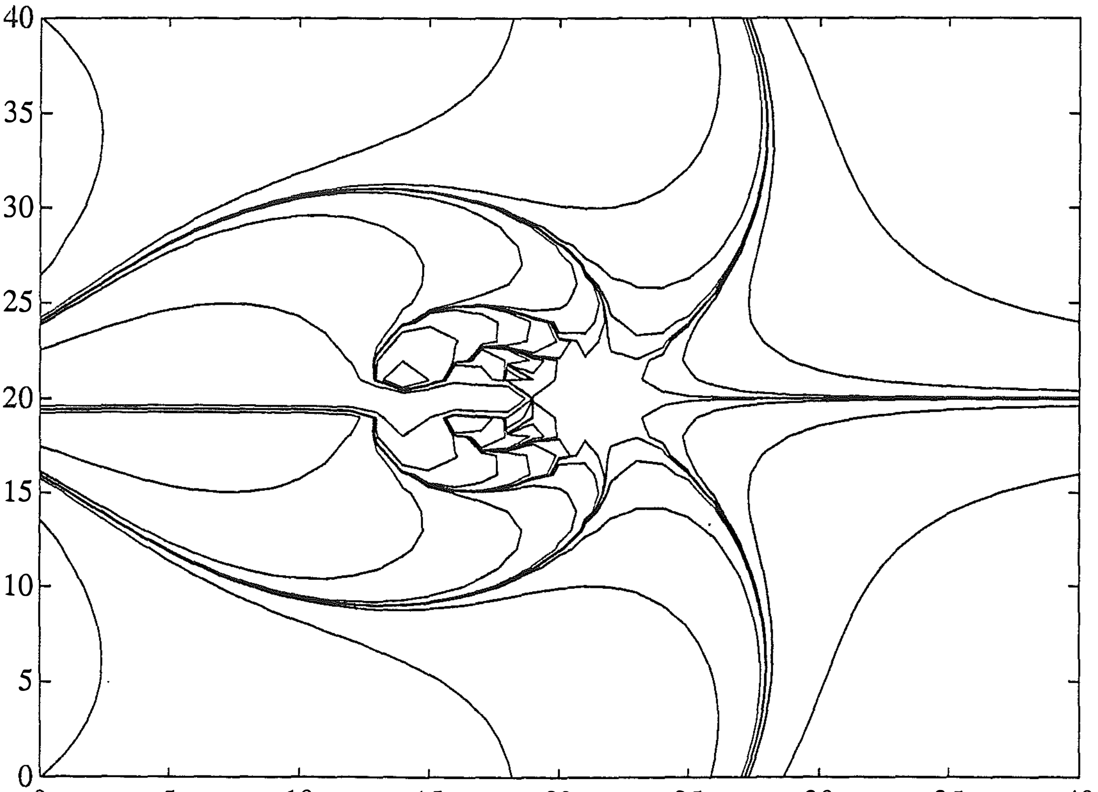

Figure 16 Contour map of Imag(e(z)).

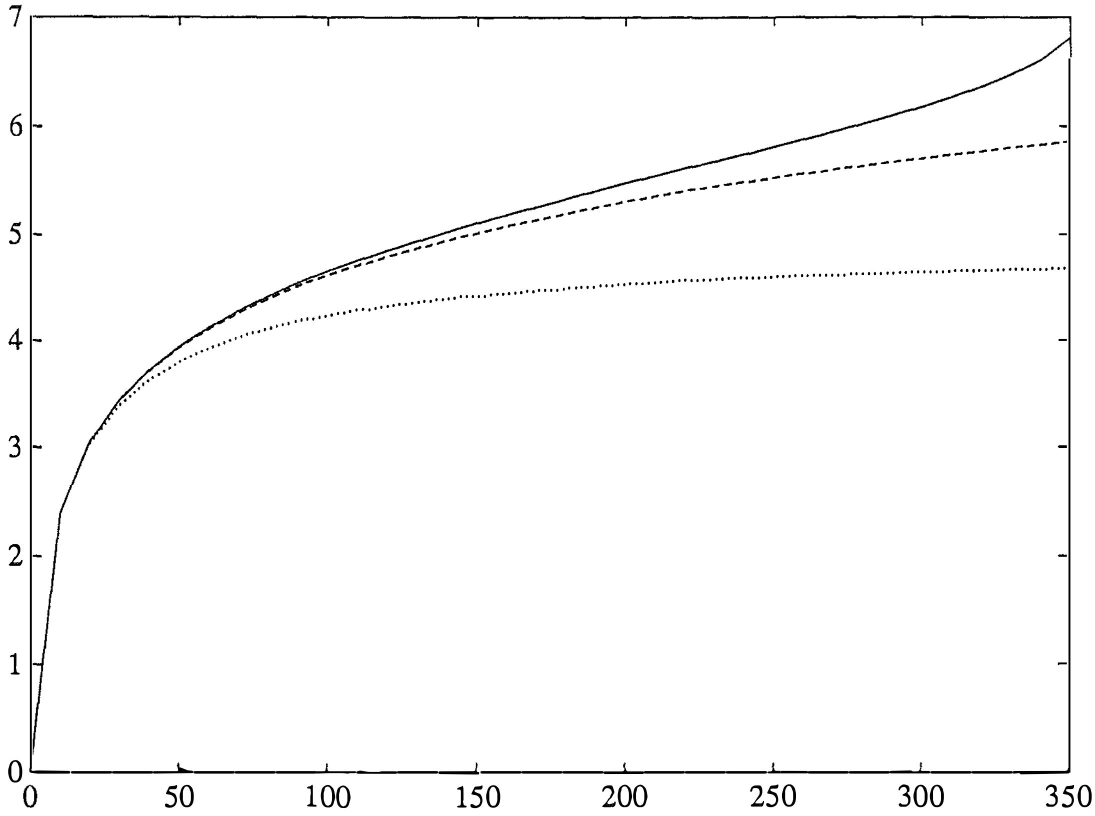

Figure 17

Example 3 : $\sqrt[3]{1+x}$.

The $(4,4,4)$ **approximation to** $\sqrt[3]{1+x}$. Note that

(i)

$$\begin{aligned}&\left(x^4-360x^3+1917x^2+2916x+729\right)f(x)^2\\&+\left(-14x^4+945x^3-513x^2-2916x-1458\right)f(x)\\&+\left(91x^4-1638x^3-2457x^2+729\right)\qquad=O\left(x^{14}\right)\end{aligned}$$

(ii)

$$y(x)=\frac{-a_1(x)+x\sqrt{d(x)}}{2a_2(x)}$$

where

$$\begin{aligned}d(x)=&-168x^6+111132x^5-2139291x^4\\&+7072758x^3+32470389x^2\\&+34720812x+11573604\,.\end{aligned}$$

The roots of $d(x)$ are:
$x=641.7609$
$x=11.2874\pm 0.0393\mathbf{i}$
$x=-0.9186\pm 0.0003\mathbf{i}$
$x = -0.9984$

Treating the roots of $d(x)$ as in the previous example and examining the region $\{x+\mathbf{i}y\in\mathbf{C}: |x|,|y|\leq 20\}$ with a mesh spacing of 1 it can be seen that $y(x)$ is a good approximation to $\sqrt[3]{1+x}$ with a similar branch point structure.

Figures 18 and 19 are graphs of real($e(z)$) and imag($e(z)$) truncated at ± 1. Figures 20 and 21 are the usual contour maps of real($e(z)$) and imag($e(z)$).

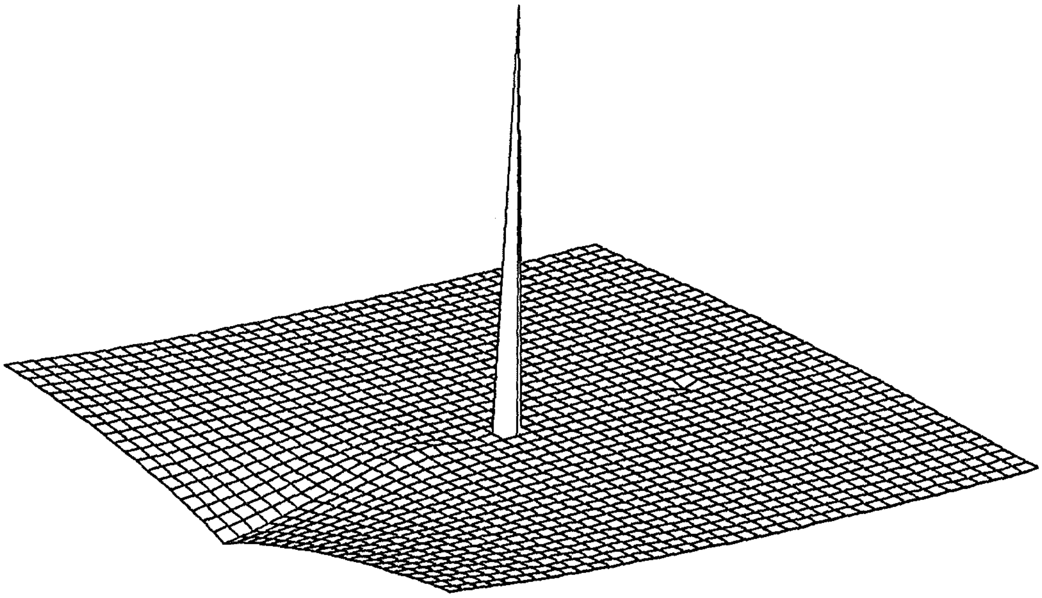

Figure 18 Real(e(z)). Truncation ± 1

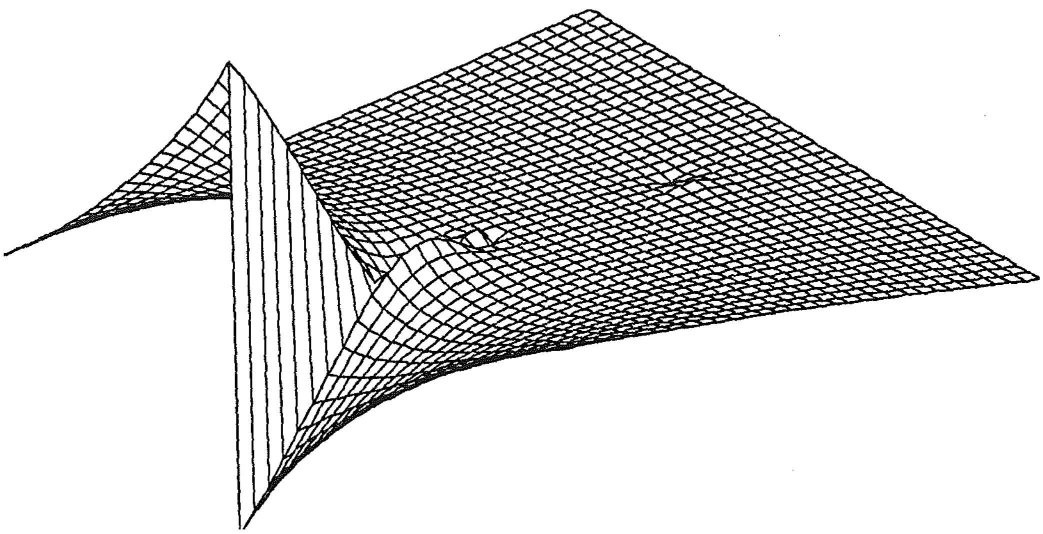

Figure 19 Imag(e(z)). Truncation ± 1

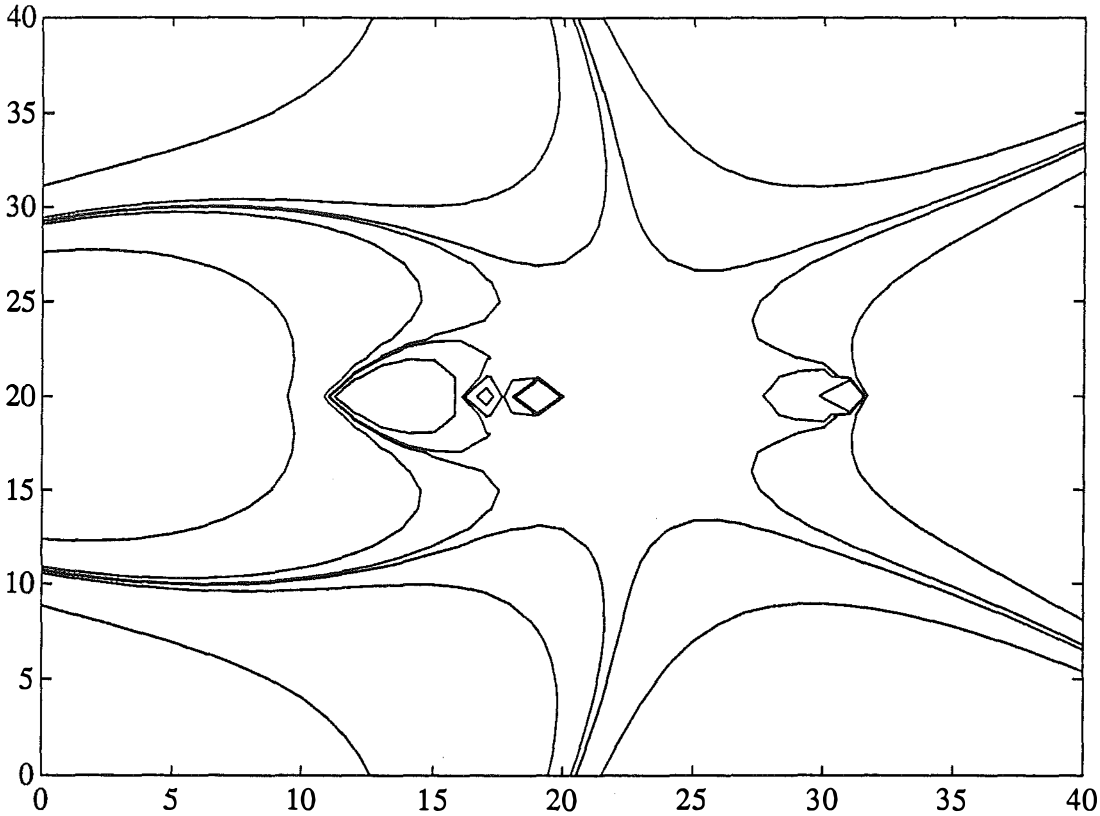

Figure 20 Contour map of Real(e(z)).

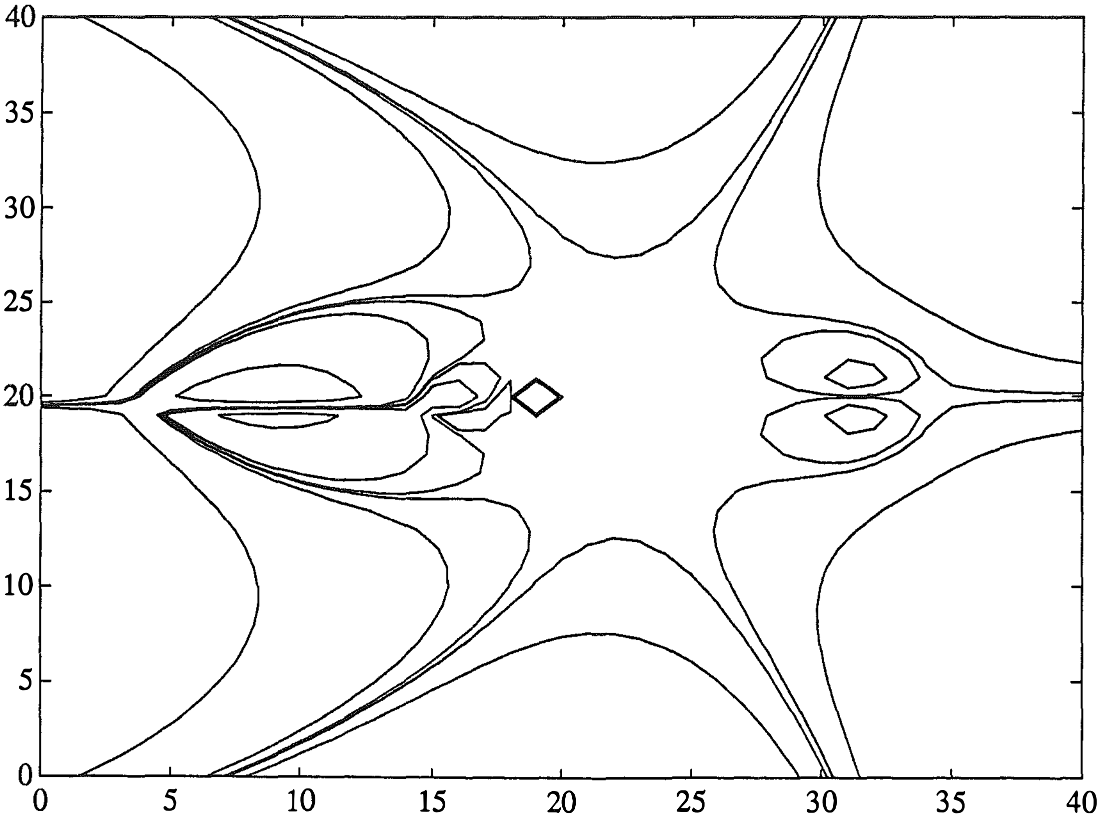

Figure 21 Contour map of Imag(e(z)).

The $(6,6)$ Padé approximation to $\sqrt[3]{1+x}$. Note that

$$p(x) = \frac{988x^6 + 31122x^5 + 266760x^4 + 960336x^3 + 1662120x^2 + 1371249x + 433026}{187x^6 + 11781x^5 + 141372x^4 + 636174x^3 + 1301265x^2 + 1226907x + 433026}$$

and that

$$y(x) = f(x) + O\left(x^{13}\right)$$
$$p(x) = f(x) + O\left(x^{13}\right) .$$

Figures 22 and 23 are graphs of real$(e(z))$ and imag$(e(z))$ truncated at ± 1. Figures 24 and 25 are the usual contour maps of real$(e(z))$ and imag$(e(z))$.

One can draw here the same conclusions as in the previous example. Again, as a further illustration a graph showing $p(x), y(x), \sqrt[3]{1+x}$ along the positive real axis from 0 to 600 is given in Figure 26. Here $y(x)$ is represented by a solid line, $\sqrt[3]{1+x}$ by "– –" and $p(x)$ by "...". Again beyond the branch point real$(y(x))$ is still a good approximation.
e.g.

$$f(999) = 10$$
$$real(y(999)) = 10.17$$
$$p(999) = 5.12$$

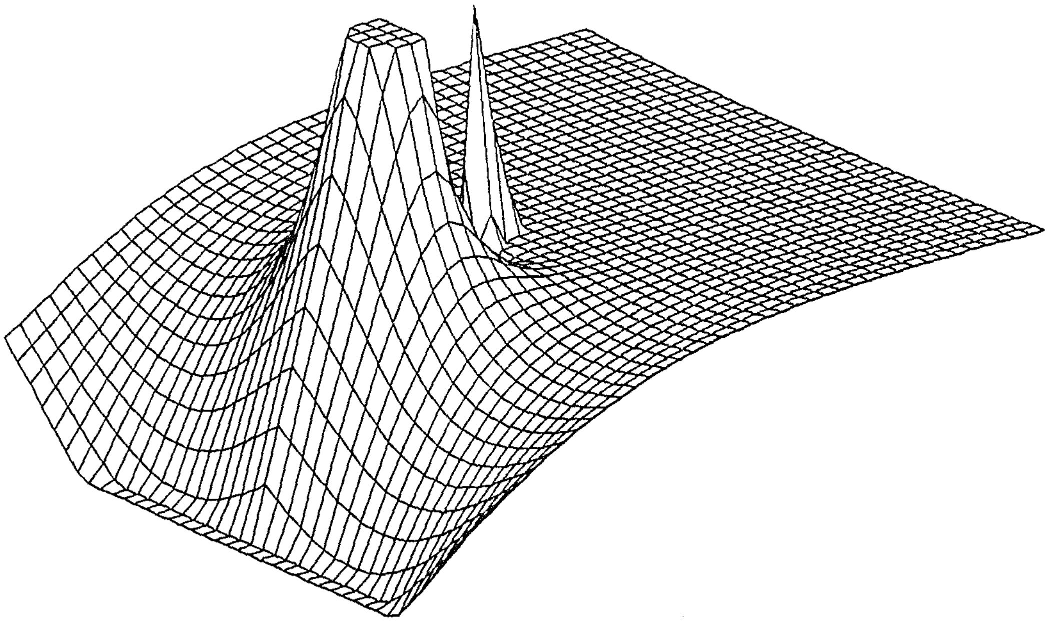

Figure 22 Real(e(z)). Truncation ± 1

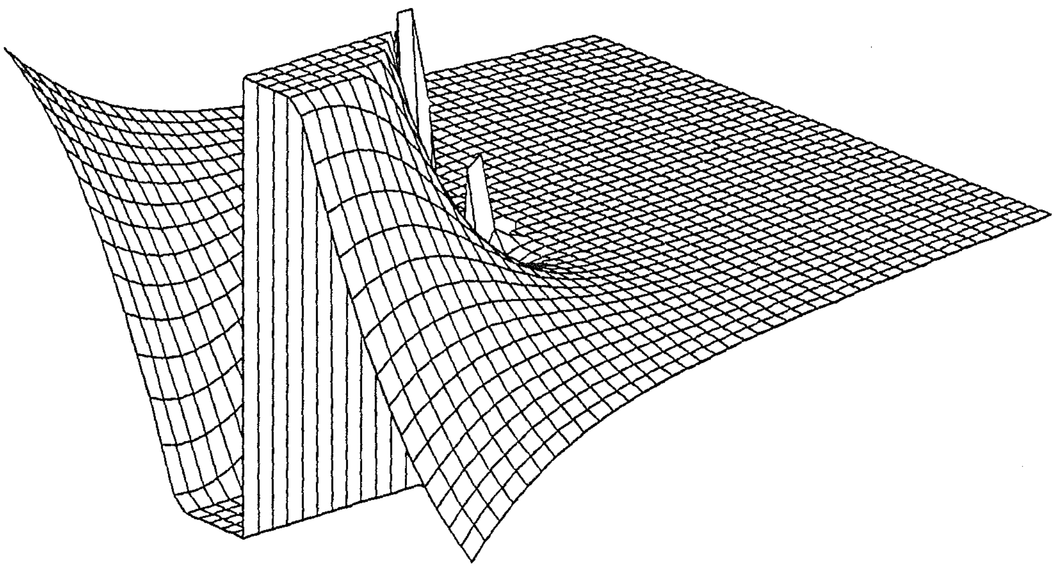

Figure 23 Imag(e(z)). Truncation ± 1

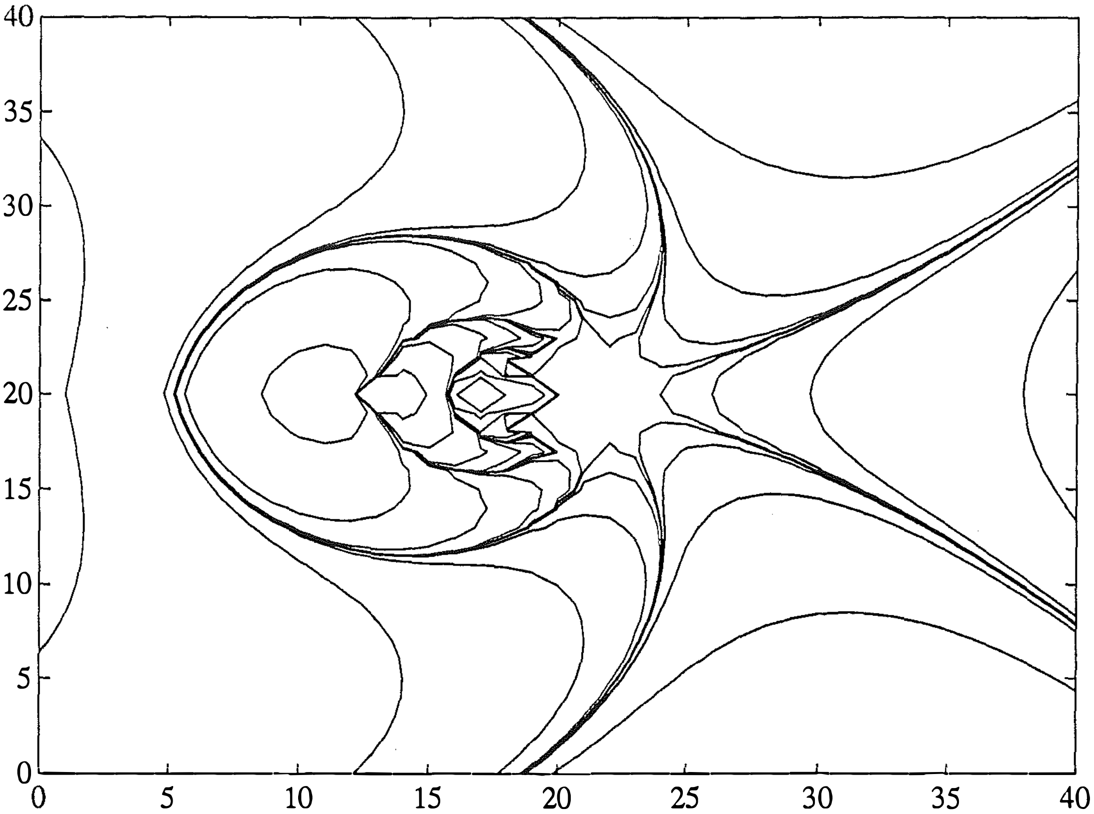

Figure 24 Contour map of Real(e(z)).

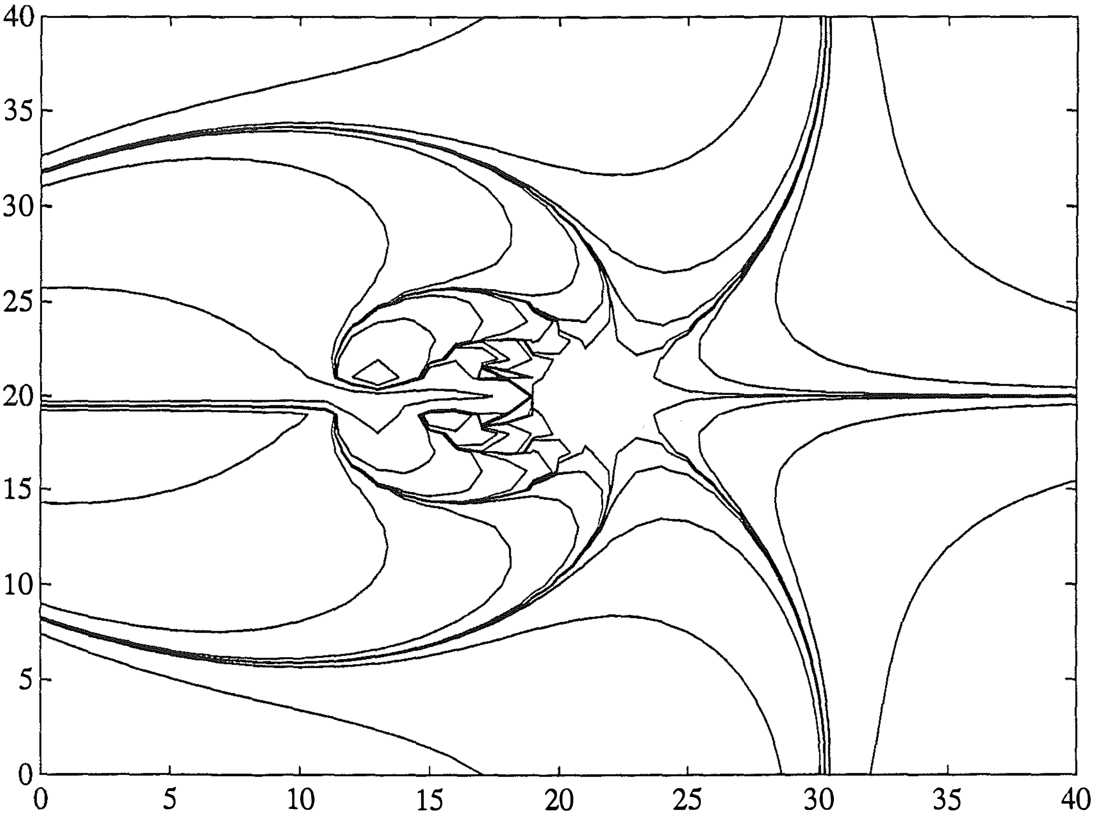

Figure 25 Contour map of Imag(e(z)).

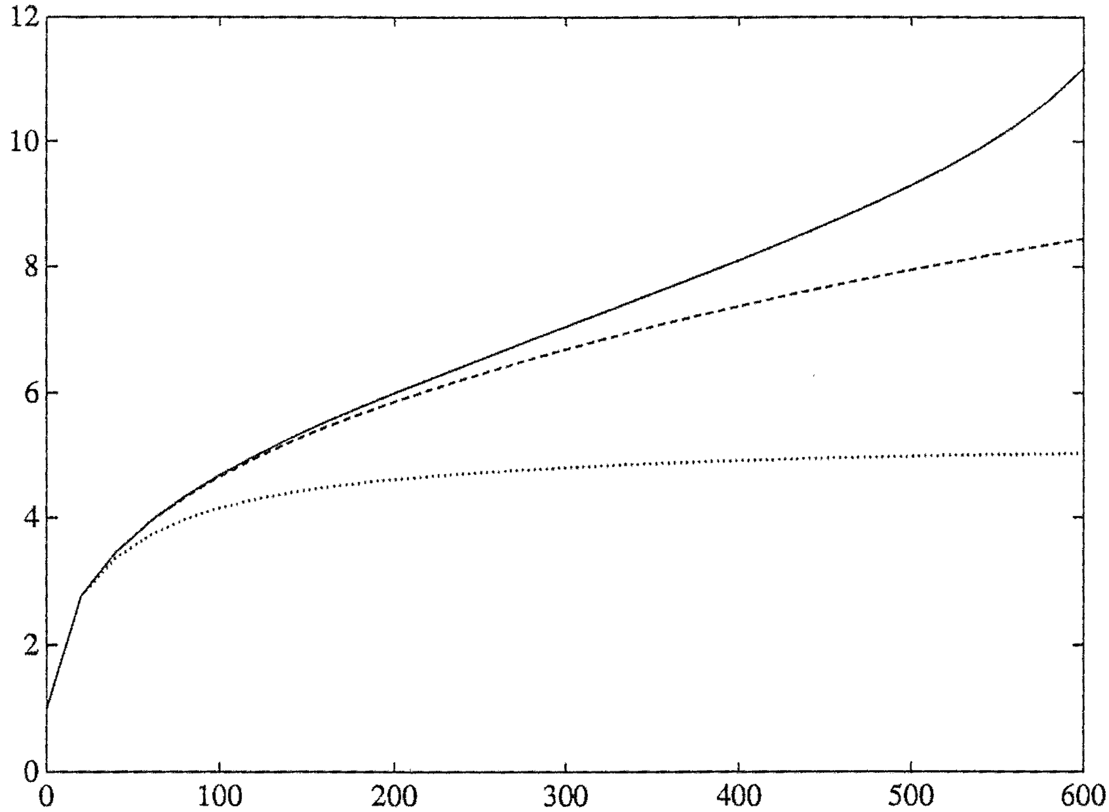

Figure 26

3. Conclusion

In each of these examples the area over which the quadratic approximation performs well seems, at first, to be limited by points at which $D(x) = 0$. If, however, these points are zeroes of "small" separation this has been shown not to be the case. The quadratic approximation may be extended beyond these points giving an approximation which is significantly better than the Padé approximation over a wide area.

References

1. R.G. Brookes, A.W. McInnes (1988): *The existence and local behaviour of the quadratic function approximation.* Math Research Report 45, Univesity of Canterbury.
2. R.G. Brookes, A.W. McInnes (1988): *Some qualitative results for the quadratic function approximation.* Math Research Report 46, University of Canterbury.